LOSS DISTRIBUTION MODELLING OF A CREDIT PORTFOLIO THROUGH EXTREME VALUE THEORY (EVT)

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Various portfolio risk models are used to calculate the probability distribution of credit losses for the issuer's portfolio with constant between-asset default correlation over a certain period of time. In absence of historical credit default data, we propose a Monte Carlo model of credit portfolios by simulating expected loan loss through extreme value theory (EVT). Hence, we attempt to solve the puzzling question of properly estimating the risk premium for expected credit loss of credit portfolios.

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1 LOSS DISTRIBUTION OF UNIFORM COLLATERAL PORTFOLIO

Past attempts to simulate credit risk of standard bank loan portfolios has been largely based on the notion that the probability of default of a uniform portfolio is consistent with a normal inverse distribution as the number of loans grows to infinity. According to VASICEK (1987), FINGER (1999) and OVERBECK AND WAGNER (2001) the normal inverse distribution $NID(p,\rho)$ with default probability $p>0$ as mean and equal pairwise asset correlation $\rho<1$ for a portfolio of h loans with equal exposure $1/h$ for $h \to +\infty$, the cumulative distribution function with p

$$NID(x,p,\rho) = N\left(\frac{1}{\sqrt{p}}\left(\sqrt{1-\rho}\,N^{-1}(x) - N^{-1}(p)\right)\right) \tag{0.1}$$

denotes the distribution of portfolio losses $0 \le x \le 1$ by drawing on the assumption on normally distributed asset returns. Its density is represented by

$$\phi(x,p,\rho) = \frac{1-\rho}{\rho} \times \frac{1}{n\left(N^{-1}(x)\right)} \times n\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,N^{-1}(x) - N^{-1}(p)\right)\right), \tag{0.2}$$

with standard deviation of the standard normal distribution function N derived from the bivariate normal distribution $N_2(x,y;\rho)$ with a zero expectation vector,[1] such that

$$\sigma = \sqrt{N_2\left(N^{-1}(p), N^{-1}(p);\rho\right) - p^2}\ . \tag{0.3}$$

2 EXTREME VALUE THEORY AS LOSS FUNCTION

Alternatively to the normal inverse distribution of random variables on a uniform space, one might resort to *extreme value theory* to model the loss density function of credit portfolios. We derive a loss function as a specialised form of a Pareto-like distribution (Fig. 26), which is one-dimensional by definition. Neither the generalised Pareto distribution (GPD) nor the transformed GDP presented in

[1] The bivariate normal distribution has a symmetric covariance matrix displaying the correlation factor ρ off and covariances on the diagonal.

this model are derived from a multi-dimensional distribution with dependent tail events (EMBRECHTS, 2000; EMBRECHTS, MCNEIL AND STRAUMANN, 1999), even though we value contingent claims on a multi-asset portfolio of securitisable loans affected by default losses. This methodology is justified on the grounds of the stochastic characteristics of the reference portfolio. Since the loan pool exhibits equal between-asset correlation, we can do without multi-dimensional distributions by considering the reference portfolio to be one asset, whose credit risk is modelled on aggregate.

Extreme value theory (EVT) propagates a stochastic methodology as part and parcel of a comprehensive risk measure to monitor asset exposure to extremes of random phenomena. EMBRECHTS (2000) describes it as a "canonical theory for the (limit) distribution of normalised maxima of independent, identically distributed random variables", where solving for the right limit results of the equation (0.4) below yields the estimation of the extremal events (EMBRECHTS, KLÜPPELBERG AND MIKOSCH, 1997; EMBRECHTS, RESNICK AND SAMORODNITSKY, 1999; MCNEIL, 1999),

$$M_n = \max\left(X_1, ..., X_n\right) \tag{0.4}$$

This is in stark opposition to the theory of averages, where

$$S_n = X_1 + ... + X_n \tag{0.5}$$

describes the general notion of quantiles as multiples of standard deviations, with the Brownian motion as a basic assumption representing what is known to be the most familiar consideration of modelling diffusion processes. Multivariate EVT as an advanced form of estimating the extreme events in a random setting (EMBRECHTS, HAAN and HUANG, 1999), purports to translating the behaviour of such rare events into stochastic processes, evolving dynamically in time and space, by considering issues such as the shape of the distribution density function (skewness and kurtosis) and its variability in stress scenarios. However, the detachment of EVT from the straightjacket of hitherto distributional assumptions on dependent tail behaviour of stochastic processes does come at a certain cost. The methodological elegance of estimating extreme events, be it normalised maxima of i.i.d. events or the behaviour thereof in the context of a stochastic process, admit to restrictions to an unreserved and unqualified adoption in credit risk management. For one, EVT features substantial intrinsic model risk (EMBRECHTS, 2000), for its requires mathematical assumptions about the tail model, whose estimation beyond or at the limit of available data defies reliable verification in practice. The absence of an optimal canonical choice of the threshold above which data is to be used imposes deliberate exogeneity on EVT modelling, which could compound limitations of the model in the presence of non-linearities (RESNIK, 1998). While these qualifications could possibly upset some of the virtues of EVT, a common caveat to EVT, nonetheless, does not hold for the presented model. High dimensional portfolios will not impair the assessment of stochastic properties of extreme events (EMBRECHTS, HAAN AND HUANG, 1999), since we model rare events of default risk in a uniform credit portfolio as a proxy for the valuation of contingent claims on defaultable multi-asset portfolios.

In a nutshell, the use of EVT as a methodology comes to matter as it best describes the stochastic behaviour of extreme events at the cost of strong distributional assumptions, for loss of less presumptive models with equal predictive power. In the context of loan securitisation the modelling of senioritised payout to investors from an underlying reference portfolio of loans over a given period of time, cases of portfolio distress do constitute extreme events in the sense of EVT. Given the objective of the proposed model to explain the effects of loss allocation and the security design provisions governing contingent claims under extreme events, EVT claims methodological attractiveness due to ease of application and flexibility in model calibration. Nevertheless, it certainly falls short of representing the ultimate panacea of risk management due to a multitude of unresolved theoretical issues, such as multiple risk factors and possible computational instability as ML estimated parameters do not necessarily converge (EMBRECHTS, 2000).

In defiance of the standard assumption of an elliptic distribution

$$f(x_j) \sim N(\mu_{x_j}, \sigma_{x_j}), \tag{0.6}$$

since a heavy upper tail of periodic credit losses x_j yields for some positive integer a

$$\int_0^\infty x_j^a f(x_j)dx_j \to +\infty, \tag{0.7}$$

the generalised Pareto distribution (GPD) with parameters $\xi \in \mathrm{R}$, $\beta > 0$ is defined by

$$G_{\xi,\beta}(x) = \begin{cases} 1 - \left(1 + \dfrac{\xi x}{\beta}\right)^{-\xi^{-1}} & \text{for } \xi \neq 0 \\ 1 - \exp\left(-\dfrac{x}{\beta}\right) & \text{for } \xi = 0 \end{cases}, \tag{0.8}$$

where $x \geq 0$ for $\xi \geq 0$ and $0 \leq x \leq -\frac{\beta}{\xi}$ for $\xi < 0$. ξ is a shape parameter of the distribution responsible for the tail behaviour, where the two cases $\xi \geq 0$ and $\xi < 0$ yield heavy tails and light tails respectively.

In order to construct a loss distribution with the same tail behaviour, we improve on the generalised Pareto distribution by following the approach introduced by JUNKER AND SZIMAYER (2001) in allowing for a peak different from zero. Hence, the following loss function *L(x)* can be derived from expanding the support of GPD to **R** by an appropriate transformation,

$$L(x) = 1 - \left(1 + \frac{\xi \times \left((x-\rho) + \sqrt{(x-\rho)^2 + s^2} \right)}{2\beta \times \left(1 + \exp\left(-\frac{\delta (x-\rho)}{\beta} \right) \right)} \right)^{-\xi^{-1}}, \tag{0.9}$$

for $\xi > 0$ (heavy tailed), $\beta > 0$, $s > 0$ and $\rho \in \mathrm{R}$ (for the treatment of $\xi \leq 0$ see JUNKER AND SZIMAYER (2001)).

Mapping of the loss function onto a distribution on the uniform interval of random variables in $[0,1]$ is achieved by imposing a upper and lower bound on *x* such that $x \in [-d; d]$ and

$$L_d(x) = \frac{L(x) - L(-d)}{L(d) - L(-d)}. \tag{0.10}$$

Subsequently, L_U is formed with $u \in [0,1]$ such that

$$L_U(u) = L_d\left(U_d^{-1}(u)\right), \tag{0.11}$$

with U_d being the uniform distribution with $\min = -d$ and $\max = d$. $\rho_u = U_d^{-1}(\rho)$ is gained through re-parameterisation, whilst β and s are scale parameters dependent on the level of d, e.g. for d' one obtains $\beta' = \beta \times \frac{d'}{d}$. The same holds true for s analogously. The following parameters have been chosen for the simulation: $\xi = 0.4, \beta = 26, s = 7.5, p_u = 10^{-4}, d = 10^4$. This parameterisation results in $1 - L(d) = 6 \cdot 10^{-7}$, which has the desired property of leaving the loss tail shape unaffected by the truncation.[2]

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[2] Since $L(-d) = 0.05$ the density of L_U does not revert to zero at point $u = 0$, which corresponds to the practical intuition of portfolio losses (reality check of uniform mapping assumption for the distribution of random variables on the uniform interval [0,1]).

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